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Coupling coefficients of $SO(n)$ and integrals involving Jacobi and Gegenbauer polynomials

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Abstract

The expressions for the coupling coefficients ($3j$-symbols) for most degenerate (symmetric) representations of orthogonal groups $SO(n)$ in a canonical basis (with $SO(n)$ restricted to $SO(n-1)$) and different semicanonical or tree bases (with $SO(n)$ restricted to $SO(n') \times SO(n'')$, $n' + n'' = n$) are considered, respectively, in context of integrals involving triplets of the Gegenbauer and the Jacobi polynomials. Since the directly derived triple-hypergeometric series do not reveal the apparent triangle conditions of the $3j$-symbols, they are rearranged, using their relation with semistretched isofactors of the second kind for the complementary chain $Sp(4) \supset SU(2) \times SU(2)$ and analogy with the stretched $9j$ coefficients of $SU(2)$, into formulae with more rich limits for summation intervals and obvious triangle conditions. The isofactors of class-one representations of orthogonal groups or class-two representations of unitary groups (and, of course, the related integrals involving triplets of the Gegenbauer and the Jacobi polynomials) turn into double sums in the cases of canonical $SO(n) \supset SO(n-1)$ or $U(n) \supset U(n-1)$ and semicanonical $SO(n) \supset SO(n-2) \times SO(2)$ chains, as well as into the ${}_4F_3(1)$ series under more specific conditions. Some ambiguities of the phase choice of the complementary group approach are adjusted, as well as problems with an alternating sign parameter of $SO(2)$ representations in the $SO(3) \supset SO(2)$ and $SO(n) \supset SO(n-2) \times SO(2)$ chains.

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1. Introduction

The coupling (Clebsch–Gordan) coefficients and $3j$-symbols (Wigner coefficients) of orthogonal groups $SO(n)$, together with their isoscalar factors (isofactors), maintain great importance in many fields of theoretical physics such as atomic, nuclear and statistical physics. The representation functions in terms of Gegenbauer (ultraspherical) polynomials are well known for symmetric (also called most degenerate or class-one) irreducible representations (irreps) of $SO(n)$ in the spherical coordinates (Vilenkin [1]) on the unit sphere S_{n-1}. In particular, the explicit Clebsch–Gordan (CG) coefficients and the isofactors of $SO(n)$ in a canonical basis for all three symmetrical irreps were considered by Gavrilik [2], Kildyushov and Kuznetsov [3] (see also [4]) and Junker [5], using the direct [2, 5] or rather complicated indirect [3, 4] integration procedures.

Norvaišas and Ališauskas [6] also derived triple-sum expressions for related isofactors of $SO(n)$ in the case of canonical (labelled by the chain of groups $SO(n) \supset SO(n-1)$) and semicanonical bases (labelled by irreps l, l', l'' of the group chains $SO(n) \supset SO(n') \times SO(n''), n' + n'' = n$, in polyspherical, or tree-type, coordinates [1, 4, 7]), exploiting the transition matrices [8] (also cf [9]) between the bases, labelled by the unitary and orthogonal subgroups in the symmetrical irreducible spaces of the $U(n)$ group. They observed [6, 10] that isofactors for the group chain $SO(n) \supset SO(n') \times SO(n'')$ for coupling of the states of symmetric irreps l_1, l_2 are the analytical continuation of the isofactors for the chain $Sp(4) \supset SU(2) \times SU(2)$,

$$\begin{bmatrix} l_1 & l_2 & [L_1 L_2] \\ l_1', l_1'' & l_2', l_2'' & \gamma [L_1' L_2'][L_1'' L_2''] \end{bmatrix}_{(n:n'n'')} = (-1)^{\phi} \begin{bmatrix} \left\langle \begin{matrix} \frac{-2L_2'-n'}{4} & \frac{-2L_1'-n'}{4} \\ \frac{-2l_1'-n'}{4}, & \frac{-2l_2'-n'}{4} \end{matrix} \right\rangle & \left\langle \begin{matrix} \frac{-2L_2''-n''}{4} & \frac{-2L_1''-n''}{4} \\ \frac{-2l_1''-n''}{4}, & \frac{-2l_2''-n''}{4} \end{matrix} \right\rangle & \left\langle \begin{matrix} \frac{-2L_2-n}{4} & \frac{-2L_1-n}{4} \\ \frac{-2l_1-n}{4}, & \frac{-2l_2-n}{4} \end{matrix} \right\rangle^{\gamma} \end{bmatrix} \quad (1.1)$$

(i.e. they coincide, up to phase factor $(-1)^{\phi}$, with the isofactors for the non-compact complementary group [11–13] chain $Sp(4, R) \supset Sp(2, R) \times Sp(2, R)$ in the case for the discrete series of irreps). In particular, in the special multiplicity-free case (for $L_2 = L_2' = L_2'' = 0$, when the label γ is absent), isofactors of $SO(n) \supset SO(n-1)$ correspond to the semistretched isofactors of the second kind [14] of $Sp(4) \supset SU(2) \times SU(2)$ (see also [15, 16]).

However, neither expressions derived by means of direct integration [2, 5], nor expressions derived by the re-expansion of states of the group chains [6, 10] reveal the apparent triangle conditions of the $3j$-symbols in these triple-sum series. Only the substitution group technique of the $Sp(4)$ or the $SO(5)$ group [17], used together with an analytical continuation procedure, enabled an indication [6, 10, 15] of the transformation of the initial triple-sum expressions of [6] into other forms, more convenient in the cases close to the stretched ones (e.g. for small values of $l_1 + l_2 - l_3$, where $l_3 = L_1$) and turning into double sums for the canonical basis, the $SO(n) \supset SO(n-2) \times SO(2)$ chain and other cases with specified parameters $l_1'' + l_2'' - l_3'' = 0$ (where $l_3'' = L_1''$). More specified isofactors of $SO(n) \supset SO(n-1)$ [16] are related to $6j$ coefficients of $SU(2)$ (with some parameters a multiple of $1/4$ in the case where n is odd).

Unfortunately, the empirical phase choices of isofactors in [6, 10, 15, 16][1] were not correlated to the basis states (cf [1, 4, 18]) in terms of the Gegenbauer and the Jacobi polynomials. Some aspects of the isofactor symmetry problem were also left untouched, e.g. the problem of the sign change for irreps m of the $SO(2)$ subgroups (which was not revealed in [1, 4, 18] for the states of $SO(3) \supset SO(2)$ and $SO(n) \supset SO(n-2) \times SO(2)$ either), as well as the indefiniteness of the type $(2l'' + n'' - 2)(n'' - 4)!!$ for $l'' = 0, n'' = 2$ in numerator or denominator.

Presentation of the unambiguous proof of the most preferable and consistent expressions for the $3j$-symbols of orthogonal $SO(n)$ and unitary $U(n)$ groups for decomposition of the factorized ultraspherical and polyspherical harmonics (i.e. for coupling of the three most

[1] Note that the phase factor $(-1)^{(g-e)/2}$ (where $g \geqslant e$) should be omitted on the right-hand side of expression (5.7) of [16] for recoupling ($6l$) coefficients of $SO(n)$, in contrast to (5.5) of the same paper.

degenerate irreps into a scalar representation in the cases for the canonical and semicanonical bases) is the main intention of this paper (cf [2–6, 16]) and is impossible without including a comprehensive review of some adjusted previous results [5, 6, 14, 15], since some references [2–4, 6, 15] may not be easily accessible nor free from misprints. However, the main goal of this paper is a strict *ab initio* rearrangement of the most symmetric (although banal) finite triple-sum series of the hypergeometric-type in the expressions of definite integrals involving triplets of the multiplied Gegenbauer and Jacobi polynomials into the less symmetric but more convenient triple (3.2*e*), double (3.6*c*) or single (3.10*b*) sum series with summation intervals depending on the triangular conditions of the corresponding $3j$-symbols.

The related triple-hypergeometric series, appearing in expressions for semistretched isoscalar factors of the second kind of the chain $Sp(4) \supset SU(2) \times SU(2)$, are considered in section 2, together with their *ab initio* rearrangement using different expressions [19] for the stretched $9j$ coefficients of $SU(2)$. (These triple-sum series may be treated as extensions of the double-hypergeometric series of Kampé de Fériet-type [20], e.g. considered by Lievens and Van der Jeugt [21].)

A well known special integral involving a triplet of the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ [22, 23] in terms of the Clebsch–Gordan coefficients of $SU(2)$ (cf [1])

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}x \left(\frac{1+x}{2}\right)^{(\beta_1+\beta_2+\beta_3)/2} \left(\frac{1-x}{2}\right)^{(\alpha_1+\alpha_2+\alpha_3)/2} \prod_{a=1}^{3} P_{k_a}^{(\alpha_a,\beta_a)}(x)$$

$$= \left(\frac{1}{2l_3+1}\prod_{a=1}^{3}\frac{(k_a+\alpha_a)!(k_a+\beta_a)!}{k_a!(k_a+\alpha_a+\beta_a)!}\right)^{1/2} C^{l_1 l_2 l_3}_{m_1 m_2 m_3} C^{l_1 l_2 l_3}_{n_1 n_2 n_3} \tag{1.2}$$

may be derived only within the framework of the angular momentum theory [24–26], when

$$l_a = k_a + \tfrac{1}{2}(\alpha_a + \beta_a) \qquad m_a = \tfrac{1}{2}(\alpha_a + \beta_a) \qquad n_a = \tfrac{1}{2}(\beta_a - \alpha_a)$$

and

$$\alpha_a = m_a - n_a \qquad \beta_a = m_a + n_a \qquad k_a = l_a - m_a$$

$$\alpha_3 = \alpha_1 + \alpha_2 \qquad \beta_3 = \beta_1 + \beta_2$$

are non-negative integers. Unfortunately, quite an elaborate expansion [3, 4] of the product of two Jacobi or Gegenbauer polynomials in terms of the third Jacobi or Gegenbauer polynomial within the framework of (1.2) gives rather complicated multiple-sum expressions for integrals involving the ultraspherical or polyspherical functions in the generic $SO(n)$ or $U(n)$ case. In section 3, the definite integrals involving triplets of the multiplied unrestrained Jacobi and Gegenbauer polynomials are initially expressed using the direct (cf [2, 5]) integration procedure as triple sums in terms of beta and gamma functions. Later they are rearranged to more convenient forms, with a fewer number of sums, or at least, with a richer structure of the summation intervals (responding to the triangular conditions of the coupling coefficients) and a better possibility of summation (especially, under definite restrictions or for some coinciding parameters).

In section 4, some normalization and phase choice peculiarities of the canonical basis states and matrix elements of the symmetric (class-one) irreducible representations of $SO(n)$ are discussed. Then we consider the corresponding expressions of $3j$-symbols and Clebsch–Gordan coefficients of $SO(n)$, factorized in terms of integrals involving triplets of the Gegenbauer polynomials (preferable in comparison to the results of [5]) and extreme (summable) $3j$-symbols, together with alternative phase systems.

In section 5, the semicanonical basis states and matrix elements of the symmetric (class-one) irreducible representations of $SO(n)$ for the restriction $SO(n) \supset SO(n') \times SO(n'')$ $(n' + n'' = n)$ are discussed. The corresponding factorized $3j$-symbols and Clebsch–Gordan coefficients, expressed in terms of integrals involving triplets of the Jacobi polynomials and extreme $3j$-symbols, are considered, together with a special approach to the $n'' = 2$ and $n' = n''$ cases.

The spherical functions for the canonical chain of unitary groups $U(n) \supset U(n-1) \times U(1) \supset \cdots \supset U(2) \times U(1) \supset U(1)$ correspond to the matrix elements of the class-two (mixed tensor) representations of $U(n)$, which include the scalar of subgroup $U(n-1)$ (see [4]). The factorized $3j$-symbols of $U(n)$, related in this case to isofactors of $SO(2n) \supset SO(2n-2) \times SO(2)$ and expressed in terms of special integrals involving triplets of the Jacobi polynomials, are considered in section 6.

In the appendix, some special cases of the triple-sum series, used in section 2, are given, shown as turning into the double- or single-sum series for some coinciding values of parameters.

2. Semistretched isoscalar factors of the second kind of $Sp(4)$ and their rearrangement

Here the irreducible representations of $Sp(4)$ will be denoted by $\langle K\Lambda\rangle$, where the pairs of parameters $K = I_{\max}$, $\Lambda = J_{\max}$ correspond to the maximal values of irreps I and J of the maximal subgroup $SU(2) \times SU(2)$ (see [14, 17, 27]) and to the irreps of $SO(5)$ with the highest weight $[L_1L_2] = [K+\Lambda, K-\Lambda]$. Below we consider the triple series shown in [14], where the following expression for the semistretched isoscalar factors of the second kind (with the coupled and resulting irrep parameters matching the condition $K_1 + K_2 = K$) for the basis labelled by the chain $Sp(4) \supset SU(2) \times SU(2)$ was derived:

$$\begin{aligned} &\begin{bmatrix} \langle K_1\Lambda_1\rangle & \langle K_2\Lambda_2\rangle & \langle K_1+K_2, \Lambda\rangle \\ I_1J_1 & I_2J_2 & IJ \end{bmatrix} \\ &\quad = (-1)^{\Lambda_1+\Lambda_2-\Lambda}\,[(2I_1+1)(2J_1+1)(2I_2+1)(2J_2+1)(2\Lambda+1)]^{1/2} \\ &\quad\times \left[\frac{\prod_{a=1}^{2}(2K_a-2\Lambda_a)!(2K_a+1)!(2K_a+2\Lambda_a+2)!}{(2K_1+2K_2-2\Lambda)!(2K_1+2K_2+1)!(2K_1+2K_2+2\Lambda+2)!}\right]^{1/2} \\ &\quad\times \left[\begin{array}{c|ccc} K_1 & \Lambda_1 & I_1 & J_1 \\ K_2 & \Lambda_2 & I_2 & J_2 \\ K_1+K_2 & \Lambda & I & J \end{array}\right]. \end{aligned} \tag{2.1}$$

Here $11j$ coefficient [14]

$$\begin{aligned} \left[\begin{array}{c|ccc} K_1 & \Lambda_1 & I_1 & J_1 \\ K_2 & \Lambda_2 & I_2 & J_2 \\ K_1+K_2 & \Lambda & I & J \end{array}\right] &= \frac{E(K_1+K_2+\Lambda, I, J)}{\prod_{a=1}^{2} E(K_a+\Lambda_a, I_a, J_a)\nabla(K_a-\Lambda_a, I_a, J_a)} \\ &\quad\times \frac{\nabla(K_1+K_2-\Lambda, I, J)}{\nabla(II_1I_2)\nabla(JJ_1J_2)\nabla(\Lambda\Lambda_1\Lambda_2)}\widetilde{\mathbf{S}}\begin{bmatrix} K_1 & I_1 & J_1 & \Lambda_1 \\ K_2 & I_2 & J_2 & \Lambda_2 \\ K_1+K_2 & I & J & \Lambda \end{bmatrix} \end{aligned} \tag{2.2}$$

(which does not belong to the $3nj$ coefficients of angular momentum theory) is expressed in terms of the triple sum $\widetilde{\mathbf{S}}[\cdots]$ and is invariant under permutations of the three right-hand columns, when the transposition of the first two rows gives the phase factor

$$(-1)^{I_1+I_2-I+J_1+J_2-J+\Lambda_1+\Lambda_2-\Lambda}. \tag{2.3}$$

In (2.2) and further we use the notation

$$\nabla(abc) = \left[\frac{(a+b-c)!(a-b+c)!(a+b+c+1)!}{(b+c-a)!}\right]^{1/2} \tag{2.4a}$$

$$= \left[\frac{\Gamma(a+b-c+1)\Gamma(a-b+c+1)\Gamma(a+b+c+2)}{\Gamma(b+c-a+1)}\right]^{1/2} \tag{2.4b}$$

and

$$E(abc) = [(a-b-c)!(a-b+c+1)!(a+b-c+1)!(a+b+c+2)!]^{1/2}. \tag{2.5}$$

Now we present different expressions of the triple sum $\widetilde{\mathbf{S}}[\cdots]$ that appear in (2.2):

$$\widetilde{\mathbf{S}}\begin{bmatrix} K_1 & j_1^1 & j_1^2 & j_1^3 \\ K_2 & j_2^1 & j_2^2 & j_2^3 \\ K_1+K_2 & j^1 & j^2 & j^3 \end{bmatrix} = \sum_{x_1,x_2,x_3} \binom{j^1+j^2+j^3-K_1-K_2}{\sum_{a=1}^3 \left(j_1^a - x_a\right) - K_1}$$
$$\times \prod_{a=1}^{3} \frac{(-1)^{x_a}\left(2j_1^a - x_a\right)!\left(j^a - j_1^a + j_2^a + x_a\right)!}{x_a!\left(j_1^a + j_2^a - j^a - x_a\right)!} \tag{2.6a}$$

$$= \left(j_1^1 - j_2^1 + j^1\right)!\left(j_1^1 + j_2^1 + j^1 + 1\right)!\left(j^2 - j_1^2 + j_2^2\right)!\left(j_1^2 + j_2^2 + j^2 + 1\right)!$$
$$\times \sum_{x_3,u,v} \frac{(-1)^{j_1^1+j_2^1-j^1+x_3+u+v}\left(2j_1^3 - x_3\right)!\left(j^3 - j_1^3 + j_2^3 + x_3\right)!}{x_3!\left(j_1^3 + j_2^3 - j^3 - x_3\right)!v!\left(j_1^1 + j_2^1 - j^1 - v\right)!\left(j_1^1 + j_2^1 + j^1 - v + 1\right)!}$$
$$\times \frac{\left(2j_2^1 - v\right)!\left(2j_1^2 - u\right)!}{\left(j_2^1 + j_2^2 + j^3 - j_1^3 - K_2 + x_3 - v\right)!u!\left(j_1^2 + j_2^2 - j^2 - u\right)!}$$
$$\times \frac{\left(j_1^1 + j_2^1 + j_1^2 + j_2^2 + j^3 - K_1 - K_2 - u - v\right)!}{\left(j_1^2 + j_2^2 + j^2 - u + 1\right)!\left(j_1^1 + j_1^2 + j_1^3 - K_1 - x_3 - u\right)!} \tag{2.6b}$$

$$= (-1)^{K_1 - j_1^1 - j_1^2 + j_1^3} \frac{\left(j_1^3 - j_2^3 + j^3\right)!}{\left(j_1^3 + j_2^3 - j^3\right)!} \prod_{a=1}^{2} \left(j^a - j_1^a + j_2^a\right)!\left(j_1^a + j_2^a + j^a + 1\right)!$$
$$\times \sum_{x_1,x_2,x_3} \prod_{a=1}^{2} \frac{\left(2j_1^a - x_a\right)!}{x_a!\left(j_1^a + j_2^a - j^a - x_a\right)!\left(j_1^a + j_2^a + j^a - x_a + 1\right)!}$$
$$\times \frac{(-1)^{x_3}\left(2j_1^3 - x_3\right)!\left(j^3 - j_1^3 + j_2^3 + x_3\right)!}{x_3!\left(j_1^3 - j_2^3 + j^3 - x_3\right)!} \binom{K_2 - j_2^1 - j_2^2 + j_2^3}{\sum_{a=1}^3 \left(j_1^a - x_a\right) - K_1}. \tag{2.6c}$$

For the rearrangement of (2.6*a*) into (2.6*c*) we used different expressions of the stretched $9j$ coefficients [19]. We transformed the double sum over x_1, x_2 in (2.6*a*) into the sum over u, v in (2.6*b*) using relation (C1*a*)–(C1*c*) of [19] and later the double sum over v, x_3 in (2.6*b*) into the sum over x_1, x_3 in (2.6*c*) using relation (C1*f*)–(C1*b*) of [19] and replacing u by x_2. (The related transformations for the double-hypergeometric series of Kampé de Fériet-type [20] are also considered by Lievens and Van der Jeugt [21].)

We see that all three summation parameters are restricted by $j_1^1 + j_1^2 + j_1^3 - K_1$ or by $j_2^1 + j_2^2 + j_2^3 - K_2$ in (2.6*a*), as well as by $j_1^1 + j_1^2 + j_1^3 - K_1$ or by $K_1 + K_2 - j^1 - j^2 + j^3$ in (2.6*c*) (respectively, by $j_2^1 + j_2^2 + j_2^3 - K_2$ or by $K_1 + K_2 + j^i - j^k - j^l$, $(i, k, l = 1, 2, 3)$ in the different versions of (2.6*c*), related by symmetries). In other cases the interval for the linear combination of summation parameters $x_1 + x_2 + x_3$ is restricted by $j^1 + j^2 + j^3 - K_1 - K_2$ in (2.6*a*), as well as by $K_2 - j_2^1 - j_2^2 + j_2^3$ in (2.6*c*) (respectively, by $K_a + j_a^i - j_a^l + j_a^k$, $(i, k, l = 1, 2, 3;$

$a = 1, 2$) in the different versions of (2.6c), related by symmetries). Hence, there are five possibilities for the completely summable expressions for $\widetilde{\mathrm{S}}[\cdots]$ and seven cases when they turn into double sums, dissimilar to nine cases, related to the stretched $9j$ coefficients [19, 24].

For our further applications, it is more convenient to write relations (2.6a)–(2.6c) (divided by $\prod_{a=1}^{3}\left(j^a - j_1^a + j_2^a\right)!\left(j^a + j_1^a - j_2^a\right)!$) in another parametrization:

$$\widetilde{\mathcal{S}}\begin{bmatrix}\alpha_0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_3,\beta_3\\ & k_1 & k_2 & k_3\end{bmatrix}$$
$$= \sum_{z_1,z_2,z_3}\begin{pmatrix}\frac{1}{2}(\alpha_0+\beta_0) - \sum_{a=1}^{3}\left[k_a + \frac{1}{2}(\alpha_a+\beta_a)\right] - 2\\ \frac{1}{2}\beta_0 - \sum_{a=1}^{3}\left(\frac{1}{2}\beta_a + z_a\right) - 1\end{pmatrix}$$
$$\times \prod_{a=1}^{3}\frac{(-1)^{z_a}(-k_a-\alpha_a)_{z_a}(-k_a-\beta_a)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{2.7a}$$
$$= \begin{pmatrix}-(2k_i+\alpha_i+\beta_i+2)\\ -(k_i+\alpha_i+1)\end{pmatrix}\sum_{z_1,z_2,z_3}(-1)^{p_i-p_i''+z_1+z_2+z_3}\begin{pmatrix}p_i''\\ p_i-z_1-z_2-z_3\end{pmatrix}$$
$$\times \frac{(k_i+1)_{z_i}(k_i+\beta_i+1)_{z_i}}{z_i!(2k_i+\alpha_i+\beta_i+2)_{z_i}}\prod_{a\neq i}\frac{(-k_a-\beta_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!}. \tag{2.7b}$$

We use the Pochhammer symbols

$$(c)_n = \prod_{k=0}^{n-1}(c+k) = \frac{\Gamma(c+n)}{\Gamma(c)}$$

and binomial coefficients, the arguments of which are non-negative integers. Here 11 parameters of (2.6a) and (2.6c) are replaced by

$$k_1 = I_1 + I_2 - I \qquad k_2 = J_1 + J_2 - J \qquad k_3 = \Lambda_1 + \Lambda_2 - \Lambda$$
$$\alpha_0 = -2K_2 - 1 \qquad \alpha_1 = -2I_2 - 1 \qquad \alpha_2 = -2J_2 - 1 \qquad \alpha_3 = -2\Lambda_2 - 1$$
$$\beta_0 = -2K_1 - 1 \qquad \beta_1 = -2I_1 - 1 \qquad \beta_2 = -2J_1 - 1 \qquad \beta_3 = -2\Lambda_1 - 1$$

with the non-negative integers

$$p_i' = \tfrac{1}{2}(\beta_j + \beta_k - \beta_i - \beta_0) \qquad p_i'' = \tfrac{1}{2}(\alpha_j + \alpha_k - \alpha_i - \alpha_0)$$
$$p_i = k_j + k_k - k_i + p_i' + p_i'' \qquad (i, j, k = 1, 2, 3)$$

and arguments of binomial coefficients, although here parameters α_j and β_j ($j = 0, 1, 2, 3$) are negative integers. Actually, expression (2.7b) may be written in three versions.

3. Integrals involving triplets of Jacobi and Gegenbauer polynomials

It is convenient for our purposes to use the two following expressions for the Jacobi polynomials (cf equation (16) of section 10.8 of [22] and chapter 22 of [23]):

$$P_k^{(\alpha,\beta)}(x) = 2^{-k}\sum_m \frac{(-k-\alpha)_m(-k-\beta)_{k-m}}{m!(k-m)!}(-1)^m(1+x)^m(1-x)^{k-m} \tag{3.1a}$$
$$= (-1)^k\sum_m \frac{(-k-\alpha)_m(k+\alpha+\beta+1)_{k-m}}{m!(k-m)!}\left(\frac{1-x}{2}\right)^{k-m} \tag{3.1b}$$

where $\alpha > -1$, $\beta > -1$.

We introduce the following expressions for the integrals involving the product of three Jacobi polynomials $P_{k_1}^{(\alpha_1,\beta_1)}(x)$, $P_{k_2}^{(\alpha_2,\beta_2)}(x)$ and $P_{k_3}^{(\alpha_3,\beta_3)}(x)$ with a measure dependent on $\alpha_0 > -1$, $\beta_0 > -1$ and integers $\alpha_a - \alpha_0 \geqslant 0$, $\beta_a - \beta_0 \geqslant 0$ ($a = 1, 2, 3$):

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}x \left(\frac{1+x}{2}\right)^{(\beta_1+\beta_2+\beta_3-\beta_0)/2} \left(\frac{1-x}{2}\right)^{(\alpha_1+\alpha_2+\alpha_3-\alpha_0)/2} \prod_{a=1}^{3} P_{k_a}^{(\alpha_a,\beta_a)}(x)$$

$$= \widetilde{\mathcal{I}}\begin{bmatrix} \alpha_0, \beta_0 & \alpha_1, \beta_1 & \alpha_2, \beta_2 & \alpha_3, \beta_3 \\ & k_1 & k_2 & k_3 \end{bmatrix} \tag{3.2a}$$

$$= (-1)^{k_1+k_2+k_3} \widetilde{\mathcal{I}}\begin{bmatrix} \beta_0, \alpha_0 & \beta_1, \alpha_1 & \beta_2, \alpha_2 & \beta_3, \alpha_3 \\ & k_1 & k_2 & k_3 \end{bmatrix} \tag{3.2b}$$

$$= \sum_{z_1,z_2,z_3} \mathrm{B}\left(1 - \tfrac{1}{2}\beta_0 + \sum_{a=1}^{3}\left(\tfrac{1}{2}\beta_a + z_a\right), 1 - \tfrac{1}{2}\alpha_0 + \sum_{a=1}^{3}\left(\tfrac{1}{2}\alpha_a + k_a - z_a\right)\right) \times \prod_{a=1}^{3} \frac{(-1)^{z_a}(-k_a-\alpha_a)_{z_a}(-k_a-\beta_a)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{3.2c}$$

$$= \sum_{z_1,z_2,z_3} \mathrm{B}\left(1 - \tfrac{1}{2}\beta_0 + \tfrac{1}{2}\sum_{a=1}^{3}\beta_a, 1 - \tfrac{1}{2}\alpha_0 + \sum_{a=1}^{3}\left(\tfrac{1}{2}\alpha_a + k_a - z_a\right)\right) \times \prod_{a=1}^{3} \frac{(-1)^{k_a}(-k_a-\alpha_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{3.2d}$$

$$= \mathrm{B}(k_i+\alpha_i+1, k_i+\beta_i+1) \sum_{z_1,z_2,z_3} (-1)^{p_i - p_i'' + z_1+z_2+z_3} \times \binom{p_i''}{p_i - z_1 - z_2 - z_3} \frac{(k_i+1)_{z_i}(k_i+\beta_i+1)_{z_i}}{z_i!(2k_i+\alpha_i+\beta_i+2)_{z_i}} \times \prod_{a=j,k;a\neq i} \frac{(-k_a-\beta_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{3.2e}$$

where the linear combinations (triangular conditions)

$$p_i' = \tfrac{1}{2}(\beta_j + \beta_k - \beta_i - \beta_0) \geqslant 0 \qquad p_i'' = \tfrac{1}{2}(\alpha_j + \alpha_k - \alpha_i - \alpha_0) \geqslant 0$$

$$p_i = k_j + k_k - k_i + p_i' + p_i'' \geqslant 0 \qquad (i, j, k = 1, 2, 3)$$

are integers. These integrals would otherwise vanish. Two first expressions (3.2*c*) and (3.2*d*) (including $(k_1+1)(k_2+1)(k_3+1)$ terms each) are straightforward to derive using expressions (3.1*a*) or (3.1*b*) and definite integrals (see equation (6.2.1) of [23]) in terms of beta functions $\mathrm{B}(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. However, vanishing of integrals (3.2*a*) under spoiled triangular conditions is only seen directly in the estimated final expression (3.2*e*), which cannot be derived in a similar manner as (3.2*c*) and (3.2*d*).

Although parameters α_a, β_a ($a = 0, 1, 2, 3$) accept the mutually excluding values in the sums $\widetilde{\mathcal{S}}[\cdots]$ and $\widetilde{\mathcal{I}}[\cdots]$, we only see the one-to-one correspondence of the analytical continuation between series (2.7*a*) and (3.2*c*), as well as between series (2.7*b*) and (3.2*e*), if the corresponding binomial coefficients of (2.7*a*) and (2.7*b*) depending on all negative (integer or half-integer) parameters are replaced in (3.2*c*) and (3.2*e*), respectively, by beta functions. The possible zeros or poles of $\widetilde{\mathcal{S}}[\cdots]$ for parameters of the definite binomial coefficients accepting negative-integer (or half-integer) values may be disregarded, if the functions $\widetilde{\mathcal{S}}[\cdots]\binom{-\alpha_0-\beta_0-2}{-\alpha_0-1}^{-1}$ and $\widetilde{\mathcal{I}}[\cdots]\mathrm{B}^{-1}(\alpha_0+1,\beta_0+1)$ are considered. Observing that the ratio of the binomial coefficients

$$\binom{-a-b-2}{-a-1}\binom{-c-d-2}{-c-1}^{-1}$$

with negative integers a, b, c, d in equations (2.7a) and (2.7b) turns into ratio of the beta functions

$$\frac{\mathrm{B}(a+1,b+1)}{\mathrm{B}(c+1,d+1)}$$

with parameters $a, b, c, d \geqslant -\frac{1}{2}$ in relation (3.2c)–(3.2e), expression (3.2e) for integral $\widetilde{\mathcal{I}}[\cdots]$ is proved. An advantage of our new expression (3.2e) is in the restriction of all three summation parameters $z_1+z_2+z_3$ by the triangular condition p_i, in contrast to (3.2c) or (3.2d). Alternatively, the linear combination of summation parameters $p_i - z_1 - z_2 - z_3 \geqslant 0$ is restricted in addition by p_i'' (or by p_i', if some symmetry is applied) only in the ith version of (2.6a). Hence, there are three cases when expressions for integral $\widetilde{\mathcal{I}}[\cdots]$ are completely summable and six cases when they turn into double sums, in addition to the double sums which appear for $k_a = 0$ $(a = 1, 2, 3)$. However, factorization of (3.2c)–(3.2e) for $\alpha_0 = \beta_0 = 0, \alpha_i = \alpha_j + \alpha_k, \beta_i = \beta_j + \beta_k$ into a product of two CG coefficients of $SU(2)$ is not straightforward to prove.

For $\alpha_0 = 0$ and $\alpha_3 = \alpha_1 + \alpha_2$, the integrals involving the product of three Jacobi polynomials (3.2e) turn into double sums (3.3a) or (3.3b)

$$\begin{aligned}
\widetilde{\mathcal{I}}\begin{bmatrix} 0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_1+\alpha_2,\beta_3 \\ & k_1 & k_2 & k_3 \end{bmatrix} &= \mathrm{B}(k_3+\alpha_1+\alpha_2+1, k_3+\beta_3+1) \\
&\quad\times \sum_{z_1,z_2} \frac{(k_3+1)_{p_3-z_1-z_2}(k_3+\beta_3+1)_{p_3-z_1-z_2}}{(p_3-z_1-z_2)!(2k_3+\alpha_1+\alpha_2+\beta_3+2)_{p_3-z_1-z_2}} \\
&\quad\times \prod_{a=1}^{2} \frac{(-k_a-\beta_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!} \qquad (3.3a)\\
&= \mathrm{B}\,(k_1+\alpha_1+1, k_1+\beta_1+1) \\
&\quad\times \sum_{z_2,z_3} \binom{k_2+\alpha_2}{z_2} \frac{(-1)^{\alpha_2-z_2}(k_2+\alpha_2+\beta_2+1-z_2)_{k_2}(-k_3-\beta_3)_{z_3}}{z_3!(k_3-z_3)!k_2!} \\
&\quad\times \frac{(k_3+\alpha_3+\beta_3+1)_{k_3-z_3}(k_1+1)_{p_1-z_2-z_3}(k_1+\beta_1+1)_{p_1-z_2-z_3}}{(p_1-z_2-z_3)!(2k_1+\alpha_1+\beta_1+2)_{p_1-z_2-z_3}} \qquad (3.3b)
\end{aligned}$$

both related to the Kampé de Fériet [20] functions $F^{2:2}_{2:1}$. It is evident that the triple series (3.2c) and (3.2d) with $\alpha_0 = 0$ may be also extended to the negative-integer values of α_2,

$$\begin{aligned}
&\widetilde{\mathcal{I}}\begin{bmatrix} 0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_3,\beta_3 \\ & k_1 & k_2 & k_3 \end{bmatrix} \\
&\quad= (-1)^{\alpha_2}\frac{(k_2+\alpha_2)!(k_2+\beta_2)!}{k_2!(k_2+\alpha_2+\beta_2)!}\widetilde{\mathcal{I}}\begin{bmatrix} 0,\beta_0 & \alpha_1,\beta_1 & -\alpha_2,\beta_2 & \alpha_3,\beta_3 \\ & k_1 & k_2+\alpha_2 & k_3 \end{bmatrix} \qquad (3.4)
\end{aligned}$$

with invariant values of p_1 and p_3. Hence, using (3.3a) for the right-hand side of (3.4), the left-hand side of (3.4) may be expressed as a double sum for $\alpha_3 = \alpha_1 - \alpha_2$ and (3.3b) may be derived after the interchange of k_1, α_1, β_1 and k_3, α_3, β_3.

The Gegenbauer (ultraspherical) polynomial $C_k^p(\cos\theta)$ may be expressed as a finite series [22, 23], or in terms of a special Jacobi polynomial (cf [22])

$$C_k^p(\cos\theta) = \sum_{m=0}^{[k/2]} \frac{(-1)^m (p)_{k-m}}{m!(k-2m)!} 2^{k-2m} \cos^{k-2m}\theta \qquad (3.5a)$$

$$= \frac{(2p)_k}{(p+1/2)_k} P_k^{(p-1/2,p-1/2)}(\cos\theta) \qquad (3.5b)$$

where $[k/2]$ is an integer part of $k/2$ and (3.5*b*) includes almost twice as many terms as (3.5*a*).

Now we may express the integrals involving the product of three Gegenbauer polynomials $C_{l_1-l'_1}^{l'_1+n/2-1}(x)$, $C_{l_2-l'_2}^{l'_2+n/2-1}(x)$ and $C_{l_3-l'_3}^{l'_3+n/2-1}(x)$ as follows:

$$\int_0^\pi \mathrm{d}\theta\,(\sin\theta)^{l'_1+l'_2+l'_3+n-2} \prod_{i=1}^3 C_{l_i-l'_i}^{l'_i+n/2-1}(\cos\theta)$$

$$= \sum_{z_1,z_2,z_3} \mathrm{B}\left(\tfrac{1}{2}(l'_1+l'_2+l'_3+n-1), \tfrac{1}{2} + \sum_{a=1}^3 \left[\tfrac{1}{2}(l_a-l'_a) - z_a\right]\right)$$

$$\times \prod_{i=1}^3 \frac{(-1)^{z_i} 2^{l_i-l'_i-2z_i} (l'_i+n/2-1)_{l_i-l'_i-z_i}}{z_i!(l_i-l'_i-2z_i)!} \qquad (3.6a)$$

$$= (-1)^{k_1+k_2+k_3} \prod_{a=1}^3 \frac{(l'_a+n/2-1)_{(l_a-l'_a+\delta_a)/2}}{(1/2)_{(l_a-l'_a+\delta_a)/2}}$$

$$\times \widetilde{\mathcal{I}}\begin{bmatrix} -\frac{1}{2}, \frac{n-3}{2} & \delta_1-\frac{1}{2}, l'_1+\frac{n-3}{2} & \delta_2-\frac{1}{2}, l'_2+\frac{n-3}{2} & \delta_3-\frac{1}{2}, l'_3+\frac{n-3}{2} \\ & \frac{1}{2}(l_1-l'_1-\delta_1) & \frac{1}{2}(l_2-l'_2-\delta_2) & \frac{1}{2}(l_3-l'_3-\delta_3) \end{bmatrix} \qquad (3.6b)$$

$$= (-1)^{(l'_j+l'_k-l'_i)/2} \mathrm{B}\left(\tfrac{1}{2}(l_i-l'_i+\delta_i+1), \tfrac{1}{2}(l_i+l'_i-\delta_i+n-1)\right)$$

$$\times \prod_{a=1}^3 \frac{(l'_a+n/2-1)_{(l_a-l'_a+\delta_a)/2}}{(1/2)_{(l_a-l'_a+\delta_a)/2}} \sum_{z_1,z_2,z_3} \binom{(\delta_j+\delta_k-\delta_i)/2}{(l_j+l_k-l_i)/2-z_1-z_2-z_3}$$

$$\times \prod_{a\neq i} \frac{(-(l_a+l'_a-\delta_a+n-3)/2)_{z_a}((l_a+l'_a+\delta_a+n)/2-1)_{(l_a-l'_a-\delta_a)/2-z_a}}{z_a!((l_a-l'_a-\delta_a)/2-z_a)!}$$

$$\times (-1)^{z_1+z_2+z_3} \frac{((l_i-l'_i-\delta_i)/2+1)_{z_i}((l_i+l'_i-\delta_i+n-1)/2)_{z_i}}{z_i!(l'_i+n/2)_{z_i}} \qquad (3.6c)$$

$$= \widetilde{\mathcal{I}}\begin{bmatrix} \overline{\alpha}_0, \overline{\alpha}_0 & l'_1+\overline{\alpha}_0, l'_1+\overline{\alpha}_0 & l'_2+\overline{\alpha}_0, l'_2+\overline{\alpha}_0 & l'_3+\overline{\alpha}_0, l'_3+\overline{\alpha}_0 \\ & l_1-l'_1 & l_2-l'_2 & l_3-l'_3 \end{bmatrix}$$

$$\times 2^{l'_1+l'_2+l'_3+n-2} \prod_{i=1}^3 \frac{(2l'_i+n-2)_{l_i-l'_i}}{(l'_i+(n-1)/2)_{l_i-l'_i}}, \qquad \left(\text{with} \quad \overline{\alpha}_0 = \tfrac{n-3}{2}\right) \qquad (3.6d)$$

where $\frac{1}{2}(l_j+l_k-l_i) \geqslant 0$ and $\frac{1}{2}(l'_j+l'_k-l'_i) \geqslant 0$ are integers. In accordance with (3.6*c*), these integrals would otherwise vanish. Expressions (3.6*a*) (cf [5]) and (3.6*d*) (cf [2]) are derived directly (using definite integrals (6.2.1) of [23] in terms of beta functions). Further (3.6*a*) is recognized as consistent with a particular case of (3.2*d*)[2] denoted by (3.6*b*) and re-expressed, in accordance with (3.2*d*) and (3.2*e*), in the most convenient form as (3.6*c*), where $\delta_1, \delta_2, \delta_3 = 0$ or 1 (in fact either $\delta_1 = \delta_2 = \delta_3 = 0$ or $\delta_a = \delta_b = 1, \delta_c = 0$) and $\frac{1}{2}(l_a - l'_a - \delta_a)$ $(a = 1, 2, 3)$ are integers.

[2] This is the reason why (3.2*d*) is introduced.

Expression (3.6a) includes $\frac{1}{8}\prod_{a=1}^{3}(l_a - l'_a - \delta_a + 2)$ terms, when (3.6d), used together with (3.2c) or (3.2d), each include $\prod_{a=1}^{3}(l_a - l'_a + 1)$ terms; otherwise the number of terms in the ith version of the most convenient formula (3.6c) never exceeds

$$A_i = (p''_i + 1)\min\left[\tfrac{1}{2}(p_i + 1)(p_i - p''_i + 2), \tfrac{1}{4}\prod_{a\neq i}(l_a - l'_a - \delta_a + 2)\right] \tag{3.7}$$

where $p''_i = \frac{1}{2}(\delta_j + \delta_k - \delta_i) = 0$ or 1, $p_i = \frac{1}{2}(l_j + l_k - l_i)$ is an integer and i, j, k is a transposition of 1, 2, 3. This number of terms decreases in comparison to (3.7) in the intermediate region

$$\tfrac{1}{2}\min(l_j - l'_j - \delta_j, l_k - l'_k - \delta_k) < p_i < \tfrac{1}{2}(l_j - l'_j - \delta_j + l_k - l'_k - \delta_k).$$

Actually, expression (3.6c) is related to the Kampé de Fériet [20] function $F^{2:2}_{2:1}$ (for $p''_i = 0$) or to the sum of two such functions (when $p''_i = 1$). Hence, after comparing three different versions of (3.6c), the rearrangement formulae of special Kampé de Fériet functions $F^{2:2}_{2:1}$ can be derived.

Now we consider more specified integrals involving several Gegenbauer polynomials. Initially, using (3.6c) with $i = 3$ and $z_2 = \delta_2 = 0$, $z_3 = \frac{1}{2}(l_1 + l' - l_3) - z_1$, we take a special integral involving the product of two Gegenbauer polynomials (where a third trivial polynomial $C_0^{l'+n/2-1}(x) = 1$ may be inserted) in terms of the summable balanced (Saalschützian) ${}_3F_2(1)$ series (cf [28, 29]) and write

$$\begin{aligned}\int_0^{\pi} &(\sin\theta)^{2l'+n-2} C_{l_i-l'}^{l'+n/2-1}(\cos\theta) C_{l'-l'}^{l'+n/2-1}(\cos\theta) C_{l_3}^{n/2-1}(\cos\theta)\,d\theta \\ &= \int_0^{\pi} (\sin\theta)^{2l'+n-2} C_{l_i-l'}^{l'+n/2-1}(\cos\theta) C_{l_3}^{n/2-1}(\cos\theta)\,d\theta \qquad (3.8a)\\ &= \frac{(-1)^{(l_3-l_1+l')/2}\pi l'!(l_1 + l' + n - 3)!}{2^{2l'+n-3}(l_1 - l')!(J' - l_1)!(J' - l_3)!\Gamma(n/2 - 1)} \\ &\quad\times \frac{\Gamma(J' - l' + n/2 - 1)}{\Gamma(l' + n/2 - 1)\Gamma(J' + n/2)} \qquad (3.8b)\end{aligned}$$

where $J' = \frac{1}{2}(l_1 + l' + l_3)$.

Using the expansion formula of the product of two Gegenbauer polynomials as the zonal spherical functions

$$\begin{aligned}C_l^p(x)C_k^p(x) = \sum_{n=|l-k|}^{l+k} &\frac{(n + p)\Gamma(g + 2p)\Gamma(g - n + p)}{\Gamma^2(p)\Gamma(g + p + 1)\Gamma(n + 2p)\Gamma(g - n + 1)} \\ &\times \frac{\Gamma(g - l + p)\Gamma(g - k + p)}{\Gamma(g - l + 1)\Gamma(g - k + 1)} C_n^p(x)\end{aligned} \tag{3.9}$$

in terms of the third polynomial of the same type (cf [1]), where $g = \frac{1}{2}(l + k + n)$ is an integer and $l + k - n$ is even, the special integral involving three Gegenbauer polynomials (with coinciding subscripts in two cases) may also be expanded in terms of integrals (3.8b) and may be presented as follows:

$$\begin{aligned}\int_0^{\pi} &(\sin\theta)^{2l'+n-2} C_{l_1-l'}^{l'+n/2-1}(\cos\theta) C_{l_2-l'}^{l'+n/2-1}(\cos\theta) C_{l_3}^{n/2-1}(\cos\theta)\,d\theta \\ &= \frac{\pi l'!}{2^{2l'+n-3}\Gamma^3(n/2 - 1)\Gamma(l' + n/2 - 1)} \\ &\quad\times \sum_{k=|l_1-l_2|+l'}^{l_1+l_2-l'} \frac{(-1)^{(l_3+l'-k)/2}(k + n/2 - 1)}{\nabla^2\,(l'/2, l_3/2 + n/4 - 1, k/2 + n/4 - 1)} \\ &\quad\times \frac{\nabla^2\left((l_1 + l' + n)/2 - 2, l_2/2 + n/4 - 1, k/2 + n/4 - 1\right)}{\nabla^2\,((l_1 - l')/2, l_2/2 + n/4 - 1, k/2 + n/4 - 1)}\end{aligned} \tag{3.10a}$$

$$= \frac{\pi l'! \prod_{a=1}^{3} \Gamma(J - l_a + n/2 - 1)}{2^{2l'+n-3}\Gamma(n/2-1)\Gamma(l'+n/2-1)\Gamma(J+n/2)} \times \sum_u \frac{(J+l'+n-3-u)!}{u!(l'-u)!(J-l_1-u)!(J-l_2-u)!(J-l_3-l'+u)!} \times \frac{(-1)^u}{\Gamma(n/2-1+u)\Gamma(l'+n/2-1-u)} \qquad (3.10b)$$

where $J = \frac{1}{2}(l_1+l_2+l_3)$ and the gamma functions under the summation sign in the intermediate formula (3.10*a*) (which is equivalent to (15) of [30]) are included in the asymmetric triangle coefficients (2.4*b*). Finally, the sum in (3.10*a*) corresponds to the very well poised ${}_7F_6(1)$ hypergeometric series (which may be rearranged using Watson's transformation formula (2.5.1) of [29] or (6.10) of [31] into balanced ${}_4F_3(1)$ hypergeometric series) or to the $6j$ coefficient

$$\begin{Bmatrix} l' + \frac{1}{2}n - 2 & \frac{1}{2}(l_1+n) - 2 & \frac{1}{2}l_1 \\ \frac{1}{2}l_3 + \frac{1}{4}n - 1 & \frac{1}{2}l_2 + \frac{1}{4}n - 1 & \frac{1}{2}l_2 + \frac{1}{4}n - 1 \end{Bmatrix} \qquad (3.11)$$

with standard (for n even) or multiples of $1/4$ parameters, in accordance with expression (C3) of the $6j$ coefficient [32] in terms of (2.4*a*). Using the most symmetric (Racah) expression [24, 25] for (3.11), the final expression (3.10*b*) with a single sum is derived. Intervals of summation are restricted by $\min(l', J-l_1, J-l_2, J-l_3)$ and, of course, (3.10*b*) coincide with result of Vilenkin [1] for $l' = 0$.

Comparing expansion (3.6*c*) of integrals involving triplets of the Gegenbauer polynomials with (3.6*d*), we may write an expression for integrals involving triplets of special Jacobi polynomials, with mutually equal superscripts,

$$\tilde{\mathcal{I}}\begin{bmatrix} \alpha_0,\alpha_0 & \alpha_1,\alpha_1 & \alpha_2,\alpha_2 & \alpha_3,\alpha_3 \\ & k_1 & k_2 & k_3 \end{bmatrix} = \frac{[1+(-1)^{p_i}]\mathrm{B}(1/2, k_i+\alpha_i+1)}{2^{k_1+k_2+k_3+\alpha_1+\alpha_2+\alpha_3-\alpha_0+2}(1/2)_{(k_j+\delta_j)/2}(1/2)_{(k_k+\delta_k)/2}} \times \sum_{z_1,z_2,z_3} (-1)^{p_i'+(k_j+\delta_j+k_k+\delta_k)/2+z_1+z_2+z_3} \binom{(\delta_j+\delta_k-\delta_i)/2}{p_i/2 - z_1 - z_2 - z_3} \times \prod_{a=j,k;a\neq i} \frac{(-k_a-\alpha_a)_{(k_a+\delta_a)/2+z_a}(\alpha_a+(k_a+\delta_a+1)/2)_{(k_a-\delta_a)/2-z_a}}{z_a!\,((k_a-\delta_a)/2-z_a)!} \times \binom{(k_i-\delta_i)/2+z_i}{z_i} \frac{(\alpha_i+(k_i-\delta_i)/2+1)_{z_i}}{(\alpha_i+3/2)_{z_i}}. \qquad (3.12)$$

Here

$$p_i = k_j + k_k - k_i + p_i' + p_i'' \qquad \tfrac{1}{2}(k_i-\delta_i) \qquad \delta_i = 0 \text{ or } 1,$$
$$p_i' = p_i'' = \tfrac{1}{2}(\alpha_j + \alpha_k - \alpha_i - \alpha_0) \qquad (i, j, k = 1, 2, 3)$$

are non-negative integers.

Comparing expansion (3.10*b*) of integrals involving more specified triplets of the Gegenbauer polynomials with (3.6*d*), we may also write an expression for integrals involving triplets of special Jacobi polynomials,

$$\widetilde{\mathcal{I}}\begin{bmatrix}\alpha_0,\alpha_0 & \alpha_1,\alpha_1 & \alpha_1,\alpha_1 & \alpha_0,\alpha_0\\ & k_1 & k_2 & k_3\end{bmatrix}$$
$$= \frac{[1+(-1)^{p_1}]2^{2\alpha_0-2}(\alpha_1-\alpha_0)!\Gamma(\alpha_1+1/2)\prod_{a=1}^{3}\Gamma(p_a/2+\alpha_0+1/2)}{\Gamma(1/2)\Gamma\left((k_1+k_2+k_3)/2+\alpha_1+3/2\right)}$$
$$\times\frac{\Gamma(\alpha_1+1+k_1)\Gamma(\alpha_1+1+k_2)\Gamma(\alpha_0+1+k_3)}{\Gamma(2\alpha_1+1+k_1)\Gamma(2\alpha_1+1+k_2)\Gamma(2\alpha_0+1+k_3)}$$
$$\times\sum_u\frac{((k_1+k_2+k_3)/2+2\alpha_1-u)!}{u!(\alpha_1-\alpha_0-u)!(p_1/2-u)!(p_2/2-u)!(p_3/2+\alpha_0-\alpha_1+u)!}$$
$$\times\frac{(-1)^u}{\Gamma(\alpha_0+1/2+u)\Gamma(\alpha_1+1/2-u)} \tag{3.13}$$

in terms of the balanced (Saalschützian) ${}_4F_3(1)$-type series [28, 29]. Here

$$p_i = k_j + k_k - k_i + 2p'_i \qquad p'_1 = p'_2 = 0 \qquad p'_3 = \alpha_1 - \alpha_0$$

are integers.

4. Canonical basis states and coupling coefficients of $SO(n)$

The canonical basis states of the symmetric (class-one) irreducible representation $l = l_{(n)}$ for the chain $SO(n) \supset SO(n-1) \supset \cdots \supset SO(3) \supset SO(2)$ are labelled by the $(n-2)$-tuple $M = (l_{(n-1)}, N) = (l_{(n-1)}, \ldots, l_{(3)}, m_{(2)})$ of integers

$$l_{(n)} \geqslant l_{(n-1)} \geqslant \cdots \geqslant l_{(3)} \geqslant |m_{(2)}|. \tag{4.1}$$

The dimension of the representation space is

$$d_l^{(n)} = \frac{(2l+n-2)(l+n-3)!}{l!(n-2)!}. \tag{4.2}$$

Special matrix elements $D_{M0}^{n,l}(g)$ of the $SO(n)$ irreducible representation $l_{(n)} = l$ with zero for the $(n-2)$-tuple $(0, \ldots, 0)$ depend only on the rotation (Euler) angles $\theta_{n-1}, \theta_{n-2}, \ldots, \theta_2, \theta_1$ (coordinates on the unit sphere S_{n-1}) and may be factorized as

$$D_{M0}^{n,l}(g) = t_{l'0}^{n,l}(\theta_{n-1})D_{N0}^{n-1,l'}(g'). \tag{4.3}$$

Here $D_{N0}^{n-1,l'}(g')$ are the matrix elements of $SO(n-1)$ irrep $l_{(n-1)} = l'$ (with coordinates on the unit sphere S_{n-2}). Special matrix elements of $SO(n)$ $(n > 3)$ irreducible representation $l_{(n)} = l$ with the $SO(n-1)$ irrep labels $l_{(n-1)} = l'$ and 0 and $SO(n-2)$ label $l_{(n-2)} = 0$ for rotation with angle θ_{n-1} in the (x_n, x_{n-1}) plane are written in terms of the Gegenbauer polynomials as follows:

$$t_{l'0}^{n,l}(\theta_{n-1}) = \left[\frac{l!(l-l')!(n-3)!(l'+n-4)!(2l'+n-3)}{l'!(l+l'+n-3)!(l+n-3)!}\right]^{1/2}$$
$$\times (n/2-1)_{l'}2^{l'}\sin^{l'}\theta_{n-1}C_{l-l'}^{l'+n/2-1}(\cos\theta_{n-1}) \tag{4.4}$$

(see [1]) and corresponds to the wavefunction $\Psi_{k,l'}^c(\theta) = \Psi_{l-l',l'}^{l+(n-3)/2}(\theta)$ of the tree technique (of the type 2b, see (2.4) of [18]) with a factor of

$$\left[\frac{\Gamma((n-1)/2)\sqrt{\pi}d_{l'}^{(n-1)}}{\Gamma(n/2)d_l^{(n)}}\right]^{1/2}$$

for appropriate normalization in the case of integration over the group volume ($0 \leqslant \theta \leqslant \pi$) with measure $\mathrm{B}^{-1}((n-1)/2, 1/2) \sin^{n-2}\theta \, \mathrm{d}\theta$. The remaining Euler angles are equal to 0 for the matrix element (4.4). In the case of $SO(3)$, we obtain

$$D^{3,l}_{m0}(\theta_2, \theta_1) = (-1)^{(l'-m)/2} t^{3,l}_{l'0}(\theta_2)\, \mathrm{e}^{\mathrm{i}m\theta_1} \qquad l' = |m| \tag{4.5}$$

in accordance with the relation [1] between the associated Legendre polynomials $P^m_l(x)$ and the special Gegenbauer polynomials $C^{m+1/2}_{l-m}(x)$ and the behaviour of $P^m_l(x)$ under the reflection of m.

The corresponding $3j$-symbols for the chain $SO(n) \supset SO(n-1) \supset \cdots \supset SO(3) \supset SO(2)$ (denoted by brackets with a simple subscript n and labelled by sets $M_a = (l'_a, N_a)$) may be factorized as follows:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ M_1 & M_2 & M_3 \end{pmatrix}_n = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n^{-1} \int_{SO(n)} \mathrm{d}g\, D^{n,l_1}_{M_1 0}(g) D^{n,l_2}_{M_2 0}(g) D^{n,l_3}_{M_3 0}(g) \tag{4.6a}$$

$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1 & l'_2 & l'_3 \end{pmatrix}_{(n:n-1)} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ N_1 & N_2 & N_3 \end{pmatrix}_{n-1}. \tag{4.6b}$$

Here the isoscalar factors of $3j$-symbols for the restriction $SO(n) \supset SO(n-1)$ are denoted by brackets with a composite subscript $(n : n-1)$ and are expressed in terms of integrals (3.6*a*) or (3.6*c*) involving triplets of the Gegenbauer polynomials,

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1 & l'_2 & l'_3 \end{pmatrix}_{(n:n-1)} &= \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n^{-1} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ 0 & 0 & 0 \end{pmatrix}_{n-1} \\ &\times \left[\frac{\Gamma((n-1)/2)}{\pi^{5/2}\Gamma(n/2)}\right]^{1/2} \prod_{a=1}^{3} \mathcal{N}^{(n:n-1)}_{l_a;l'_a,\delta_a} \left[\frac{d^{(n-1)}_{l'_a}}{d^{(n)}_{l_a}}\right]^{1/2} \\ &\times \int_0^\pi \mathrm{d}\theta\, (\sin\theta)^{l'_1+l'_2+l'_3+n-2} \prod_{i=1}^{3} C^{l'_i+n/2-1}_{l_i-l'_i}(\cos\theta) \end{aligned} \tag{4.7}$$

where

$$\mathcal{N}^{(n:n-1)}_{l_a;l'_a,\delta_a} = 2^{l'_a+n/2-2}\Gamma(l'_a+n/2-1)\left[\frac{(l_a-l'_a)!(2l_a+n-2)}{(l_a+l'_a+n-3)!}\right]^{1/2} \tag{4.8}$$

are normalization factors and particular $3j$-symbols

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n &= (-1)^{\psi_n} \frac{1}{\Gamma(n/2)} \left[\frac{(J+n-3)!}{(n-3)!\Gamma(J+n/2)}\right. \\ &\times \left.\prod_{i=1}^{3} \frac{(l_i+n/2-1)\,\Gamma(J-l_i+n/2-1)}{d^{(n)}_{l_i}(J-l_i)!}\right]^{1/2} \end{aligned} \tag{4.9}$$

(vanishing for $J = \frac{1}{2}(l_1+l_2+l_3)$ half-integer) are derived in [5] (see also special Clebsch–Gordan coefficients [3, 6]). Equation (4.7) together with (3.6*a*) is equivalent to the result of [5], but its most convenient form is obtained when the special integral is expressed by means of a double-sum expression (3.6*c*) (for $i = 1, 2$ or 3, minimizing (3.7)), which ensure its finite rational structure for a fixed shift $\frac{1}{2}(l_1+l_2-l_3)$ of parameters. In the case of $l'_i = 0$, expression (3.10*b*) for the special integral is more convenient, in accordance with [16].

In (4.9), $J - l_i$ $(i = 1, 2, 3)$ and J are non-negative integers and $\psi_3 = J$, in accordance with angular momentum theory [24, 25]. We may also take

$$\psi_n = J \tag{4.10}$$

(see [5]) for $n \geqslant 4$ in order to obtain the isofactors (4.7) positive where the maximal values of parameters $l_1' = l_1, l_2' = l_2, l_3' = l_3$.

Only by taking into account the phase factor $(-1)^{(l'-m)/2}$ of (4.5), can we obtain the consistent signs of the usual Wigner coefficients ($3j$-symbols)

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{pmatrix}_{(3:2)} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{4.11}$$

of $SO(3)$ or $SU(2)$ (where $l_a' = |m_a|$ and $m_1 + m_2 + m_3 = 0$), with

$$\begin{pmatrix} l_1' & l_2' & l_3' \\ 0 & 0 & 0 \end{pmatrix}_2 = \delta_{\max(l_1', l_2', l_3'), (l_1'+l_2'+l_3')/2} (-1)^{(l_1'+l_2'+l_3')/2} \tag{4.12}$$

consequently appearing, in (4.7) for $n = 3$.[3]

We may write (cf [5]) the following dependence between special Clebsch–Gordan coefficients (denoted by square brackets with subscript) and the $3j$-symbols of $SO(n)$:

$$\begin{aligned} \begin{bmatrix} l_1 & l_2 & l_3 \\ M_1 & M_2 & M_3 \end{bmatrix}_n \begin{bmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{bmatrix}_n &\equiv \langle l_1 M_1; l_2 M_2 | (l_1 l_2) l_3 M_3 \rangle_n \langle (l_1 l_2) l_3 0 | l_1 0; l_2 0 \rangle_n \\ &= d_l^{(n)} \int_{SO(n)} \mathrm{d}g\, D^{n,l_1}_{M_1 0}(g) D^{n,l_2}_{M_2 0}(g) \overline{D^{n,l_3}_{M_3 0}(g)} \qquad (4.13a) \\ &= d_l^{(n)} (-1)^{l_3 - m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ M_1 & M_2 & \overline{M_3} \end{pmatrix}_n \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n \qquad (4.13b) \end{aligned}$$

where the $(n-2)$-tuple $\overline{M_3}$ is obtained from the $(n-2)$-tuple M_3 after reflection of the last parameter m_3. Then in the phase system with $\psi_n = J$, we obtain the following relation for the isofactors of CG coefficients in the canonical basis:

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{bmatrix}_{(n:n-1)} = (-1)^{l_3 - l_3'} \left[\frac{d^{(n)}_{l_3}}{d^{(n-1)}_{l_3'}} \right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{pmatrix}_{(n:n-1)} \tag{4.14}$$

which, together with (4.7), (4.9), (4.10) and (3.6c) or (3.6b) substituted by (3.2c), allows us to obtain expressions for isofactors of $SO(n) \supset SO(n-1)$ derived in [6] and satisfying the same phase conditions.

However, as was noted in [10], the choice (4.10) of ψ_n does not give the correct phases for special isofactors of $SO(4)$ [33] in terms of $9j$ coefficients of $SU(2)$ [24] for isofactors of $SO(5) \supset SO(4)$, as considered in [14, 17, 27]. The contrast of the phases is caused by the fact that the signs of the matrix elements of infinitesimal operators

$$A_{k,k-1} = x_k \frac{\partial}{\partial x_{k-1}} - x_{k-1} \frac{\partial}{\partial x_k} \qquad k = 3, \ldots, n$$

(with the exception of $A_{2,1}$) between the basis states [1] of $SO(n)$ in terms of Gegenbauer polynomials (in x_k/r_k, $r_k^2 = x_1^2 + \cdots + x_k^2$ variables) are opposite to the signs of the standard (Gel'fand–Tsetlin) matrix elements [34–36]. We eliminate this difference of phases and our results match with the isofactors for decomposition of the general and vector irreps $m_n \otimes 1$ of

[3] Of course, in this case the usual expressions [1, 24, 25] of the Clebsch–Gordan or Wigner coefficients of $SU(2)$ are more preferable in comparison to equation (4.7).

$SO(n)$ [36, 37] (specified also in [38, 39]) after we multiply isofactors of CG coefficients for the restriction $SO(n) \supset SO(n-1)(n \geqslant 4)$, i.e. the left-hand side of (4.14), by

$$(-1)^{(l_1+l_2-l_3-l'_1-l'_2+l'_3)/2}$$

(cf [10]), i.e. after we omit the phase factors $(-1)^{\psi_n}$ and $(-1)^{\psi_{n-1}}$ in both the auxiliary $3j$-symbols of (4.7) and $(-1)^{l_3-l'_3}$ in relation (4.14), again keeping the isofactors (4.14) with the maximal values of parameters $l'_1 = l_1, l'_2 = l_2, l'_3 = l_3$ for this restriction positive. In the both phase systems of the factorized $SO(n)$ CG coefficients ($3j$-symbols) the last factors coincide with the usual CG coefficients ($3j$-symbols) of angular momentum theory [24, 25].

5. Semicanonical bases and coupling coefficients of $SO(n)$

Furthermore, going to the semicanonical basis of the symmetric (class-one) irreducible representation l for the chain $SO(n) \supset SO(n') \times SO(n'') \supset SO(n'-1) \times SO(n''-1) \supset \cdots$, respectively, we introduce special matrix elements $D^{n:n',n'';l}_{l'M',l''M'';0}(g)$ depending only on the rotation angles $\theta'_{n'-1}, \ldots, \theta'_1$ and $\theta''_{n''-1}, \ldots, \theta''_1$ of subgroups $SO(n')$ and $SO(n'')$ and the rotation angle θ_c in the $(x_n, x_{n'})$ plane, with the second matrix index taken to be zero as the $(n-2)$-tuple $(0, \ldots, 0)$ for the scalar of $SO(n-1)$. These matrix elements may be factorized as follows:

$$D^{n:n',n'';l}_{l'M',l''M'';0}(g) = t^{(n)l}_{(n')l'0,(n'')l''0;(n-1)0}(\theta_c) D^{n',l'}_{M'0}(g') D^{n'',l''}_{M''0}(g''). \tag{5.1}$$

Instead of the wavefunction $\Psi^{b,a}_{k,l'',l'}(\theta_c) = \Psi^{l''+n''/2-1,l'+n'/2-1}_{(l-l'-l'')/2,l'',l'}(\theta_c)$ (of the type 2c, see (2.6) of [18]) of the tree technique after renormalization with factor

$$\left[\frac{\Gamma(n'/2)\Gamma(n''/2)d^{(n')}_{l'}d^{(n'')}_{l''}}{2\Gamma(n/2)d^{(n)}_{l}}\right]^{1/2}$$

for the integration over the group volume $(0 \leqslant \theta_c \leqslant \pi/2)$ with measure

$$2\mathrm{B}^{-1}(n'/2, n''/2) \sin^{n''-1}\theta_c \cos^{n'-1}\theta_c \,\mathrm{d}\theta_c$$

we obtain special matrix elements of the $SO(n)$ irreducible representation l in terms of the Jacobi polynomials

$$t^{(n)l}_{(n')l'0,(n'')l''0;(n-1)0}(\theta_c) = (-1)^{\varphi_{n'n''}}\left[\frac{d^{(n')}_{l'}d^{(n'')}_{l''}\Gamma(n/2)}{d^{(n)}_{l}\Gamma(n'/2)\Gamma(n''/2)}\right]^{1/2}\mathcal{N}^{(n:n',n'')}_{l:l',l''}$$
$$\times \sin^{l''}\theta_c \cos^{l'}\theta_c P^{(l''+n''/2-1,l'+n'/2-1)}_{(l-l'-l'')/2}(\cos 2\theta_c) \tag{5.2}$$

where the left-hand $SO(n') \times SO(n'')$ labels are l', l'' $(n' + n'' = n)$, the left-hand $SO(n'-1) \times SO(n''-1)$ and right-hand $SO(n-1)$ labels are 0 for rotation with angle θ_c in the $(x_n, x_{n'})$ plane. Here phase $\varphi_{n'n''} = 0$, unless $n'' = 2$ or $n' = 2$, when the left-hand side should be replaced, respectively, by $t^{(n)l}_{(n-2)l'0,(2)m'';(n-1)0}(\theta_c)$ with $l'' = |m''|$ or by $t^{(n)l}_{(2)m',(n-2)l''0;(n-1)0}(\theta_c)$ with $l' = |m'|$ and

$$\varphi_{n'n''} = \tfrac{1}{2}[\delta_{n''2}(l'' - m'') + \delta_{n'2}(l' - m')]$$

on the right-hand side and normalization factor

$$\mathcal{N}^{(n:n',n'')}_{l:l',l''} = \left[\frac{(l+n/2-1)((l-l'-l'')/2)!\Gamma((l+l'+l''+n-2)/2)}{\Gamma((l-l'+l''+n'')/2)\Gamma((l+l'-l''+n')/2)}\right]^{1/2}. \tag{5.3}$$

The $3j$-symbols for the chain $SO(n) \supset SO(n') \times SO(n'') \supset SO(n'-1) \times SO(n''-1) \supset \cdots$, labelled by the sets $M_i = (l'_i, N'_i; l''_i, N''_i)$ may be factorized as follows:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ M_1 & M_2 & M_3 \end{pmatrix}_n = \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{pmatrix}_{(n:n'n'')} \times \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ N'_1 & N'_2 & N'_3 \end{pmatrix}_{n'} \begin{pmatrix} l''_1 & l''_2 & l''_3 \\ N''_1 & N''_2 & N''_3 \end{pmatrix}_{n''}. \tag{5.4}$$

Now the $SO(n) \supset SO(n') \times SO(n'')$ isofactor of $3j$-symbol is expressed as follows:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{pmatrix}_{(n:n'n'')} = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n^{-1} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ 0 & 0 & 0 \end{pmatrix}_{n'} \times \begin{pmatrix} l''_1 & l''_2 & l''_3 \\ 0 & 0 & 0 \end{pmatrix}_{n''} \prod_{a=1}^{3} \mathcal{N}^{(n:n',n'')}_{l_a;l'_a,l''_a} \left[\frac{d^{(n')}_{l'_a} d^{(n'')}_{l''_a}}{d^{(n)}_{l_a}} \right]^{1/2} \times \mathrm{B}^{1/2}(n'/2, n''/2) \widetilde{\mathcal{I}} \begin{bmatrix} \alpha_0, \beta_0 & \alpha_1, \beta_1 & \alpha_2, \beta_2 & \alpha_3, \beta_3 \\ & k_1 & k_2 & k_3 \end{bmatrix} \tag{5.5}$$

in terms of auxiliary $3j$-symbols (4.9) of the canonical bases (turning into phase factors of the type (4.12) for $n' = 2$ or $n'' = 2$), normalization factors (5.3) and the integrals involving triplets of Jacobi polynomials (3.2a)–(3.2e), with parameters

$$k_i = \tfrac{1}{2}(l_i - l'_i - l''_i) \qquad \alpha_i = l''_i + n''/2 - 1 \qquad \beta_i = l'_i + n'/2 - 1$$
$$\alpha_0 = n''/2 - 1 \qquad \beta_0 = n'/2 - 1$$

and

$$p'_i = \tfrac{1}{2}(l'_j + l'_k - l'_i) \qquad p''_i = \tfrac{1}{2}(l''_j + l''_k - l''_i)$$
$$p_i = \tfrac{1}{2}(l_j + l_k - l_i) \qquad (i, j, k = 1, 2, 3).$$

The number of terms in expansion (3.2e) of integrals involving triplets of Jacobi polynomials never exceeds

$$B_i = \min\left(\tfrac{1}{6}(p_i + 1)_3, (p''_i + 1)(k_j + 1)(k_k + 1), \tfrac{1}{2}(p''_i + 1)(p_i + 1)_2)\right) \tag{5.6a}$$

and decreases in the intermediate region (e.g. when $p''_i < p_i + 1$), described by the volume of the obliquely truncated rectangular parallelepiped of $(p''_i + 1) \times (k_j + 1) \times (k_k + 1)$ size.

In particular, in the case of $n'' = 2$ parameters l''_1, l''_2, l''_3 in $3j$-symbols (5.5) should be replaced by $m''_1 = \pm l''_1, m''_2 = \pm l''_2, m''_3 = \pm l''_3$ such that $m''_1 + m''_2 + m''_3 = 0$. Hence at least one parameter $p''_{i'} = 0$ and the number of terms in the i'th double-sum version of (3.2e) (related to (3.3a) and to the Kampé de Fériet [20] function $F^{2:2}_{2:1}$) does not exceed

$$\widetilde{B}_{i'} = \min\left(\tfrac{1}{2}(p_{i'} + 1)_2, (k_{j'} + 1)(k_{k'} + 1)\right) \tag{5.6b}$$

although for small values of p_i such that $B_i < \widetilde{B}_{i'}$ the ith version of (3.2e) or (3.3b) may be more preferable.

We may also express the isofactors of the CG coefficients for the restriction $SO(n) \supset SO(n') \times SO(n'')$ in terms of the isofactors of $3j$-symbols,

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{bmatrix}_{(n:n'n'')} = (-1)^{\varphi} \left[\frac{d^{(n)}_{l_3}}{d^{(n')}_{l'_3} d^{(n'')}_{l''_3}} \right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{pmatrix}_{(n:n'n'')} \tag{5.7}$$

with the phase $\varphi = 0$ (since $l_3 - l'_3 - l''_3$ is even), when $\psi_n, \psi_{n'}, \psi_{n''}$ are taken to be equal to J, J', J'', respectively, in all the auxiliary $3j$-symbols (4.9), in contrast to

$$\varphi = m''_3 \delta_{n''2} + m'_3 \delta_{n'2} + l''_3 \delta_{n''3} + l'_3 \delta_{n'3}$$

appearing when ψ_n is taken to be zero for $n \geqslant 4$. Again we need to replace, respectively, for $n'' = 2$ parameters l_1'', l_2'', l_3'' on the left-hand side by m_1'', m_2'', m_3'' such that $l_1'' = |m_1''|, l_2'' = |m_2''|, l_3'' = |m_3''|$ (with $m_1'' + m_2'' = m_3''$) and on the right-hand side by $m_1'', m_2'', -m_3''$, as well as for $n' = 2$ parameters l_1', l_2', l_3' on the left-hand side by m_1', m_2', m_3' such that $l_1' = |m_1'|, l_2' = |m_2'|, l_3' = |m_3'|$ $(m_1' + m_2' = m_3')$ and on the right-hand side by $m_1', m_2', -m_3'$.

Regarding the different triple-sum versions (3.6a)–(3.6d) of integrals involving triplets of Gegenbauer and Jacobi polynomials and comparing expressions (5.5) and (4.7), we derive the following duplication relation between the generic $SO(n) \supset SO(n-1)$ and special $SO(2n+2) \supset SO(n-1) \times SO(n-1)$ isofactors of the $3j$-symbols:

$$\begin{pmatrix} 2l_1 & 2l_2 & 2l_3 \\ l_1', l_1' & l_2', l_2' & l_3', l_3' \end{pmatrix}_{(2n-2:n-1,n-1)} = \begin{pmatrix} 2l_1 & 2l_2 & 2l_3 \\ 0 & 0 & 0 \end{pmatrix}_{2n-2}^{-1} \begin{pmatrix} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{pmatrix}_{(n:n-1)} \times \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_{n} \begin{pmatrix} l_1' & l_2' & l_3' \\ 0 & 0 & 0 \end{pmatrix}_{n-1} \prod_{a=1}^{3} \left[\frac{d_{l_a}^{(n)} d_{l_a'}^{(n-1)}}{d_{2l_a}^{(2n-2)}} \right]^{1/2} \tag{5.8}$$

with auxiliary $3j$-symbols (4.9) of the canonical bases and the irrep dimensions appearing.

6. Basis states and coupling coefficients of the class-two (mixed tensor) representations of $U(n)$

Mixed tensor irreducible representations $[p+q, q^{n-2}, 0] \equiv [p, \dot{0}, -q]$ of $U(n)$ containing scalar irrep $[q^{n-1}] \equiv [\dot{0}]$ of subgroup $U(n-1)$ (with repeating zeros denoted by $\dot{0}$) are called class-two irreps [40, 41]; their canonical basis states for the chain $U(n) \supset U(n-1) \times U(1) \supset \cdots \supset U(2) \times U(1) \supset U(1)$ are labelled by the set

$$Q_{(n)} = (p_{(n-1)}, q_{(n-1)}; Q_{(n-1)}) = (p_{(n-1)}, q_{(n-1)}; p_{(n-2)}, q_{(n-2)}; \ldots, p_{(2)}, q_{(2)}; p_{(1)})$$

where

$$p = p_{(n)} \geqslant p_{(n-1)} \geqslant \cdots \geqslant p_{(2)} \geqslant 0 \qquad \text{and} \qquad q = q_{(n)} \geqslant q_{(n-1)} \geqslant \cdots \geqslant q_{(2)} \geqslant 0$$

are integers, with $p_{(2)} \geqslant p_{(1)} \geqslant -q_{(2)}$ in addition, and parameters

$$M_{(1)} = p_{(1)}, M_{(2)} = p_{(2)} - q_{(2)} - p_{(1)}, \ldots, M_{(r)} = p_{(r)} - q_{(r)} - p_{(r-1)} + q_{(r-1)}$$

which correspond to irreps of subgroups $U(1)$, beginning from the last one.

The dimension of representation space is

$$d_{[p,\dot{0},-q]}^{(n)} = \frac{(p+q+n-1)(p+1)_{n-2}(q+1)_{n-2}}{(n-1)!(n-2)!}. \tag{6.1}$$

Special matrix elements $D_{Q_{(n)};0}^{n[p,\dot{0},-q]}(g)$ of $U(n)$ irrep $[p, \dot{0}, -q]$ with zero as the second index for the scalar of subgroup $U(n-1)$ depend only on the rotation angles $\varphi_n, \varphi_{n-1}, \ldots, \varphi_2, \varphi_1$, where $0 \leqslant \varphi_i \leqslant 2\pi$ corresponds to the ith diagonal subgroup $U(1)$ $(i = 1, 2, \ldots, n)$, and $\theta_n, \theta_{n-1}, \ldots, \theta_3, \theta_2$, where $0 \leqslant \theta_r \leqslant \pi/2$, corresponds to the transformation

$$\begin{vmatrix} \cos\theta_r & \mathrm{i}\sin\theta_r \\ \mathrm{i}\sin\theta_r & \cos\theta_r \end{vmatrix}$$

in the plane of $(r-1)$st and rth coordinates $(r = 2, 3, \ldots, n)$ and may be factorized as follows:

$$D_{Q_{(n)};0}^{n[p,\dot{0},-q]}(g) = \mathrm{e}^{\mathrm{i}M_n\varphi_n} D_{[p',\dot{0},-q']0;0}^{n[p,\dot{0},-q]}(\theta_n) D_{Q_{(n-1)};0}^{n-1[p',\dot{0},-q']}(g') \tag{6.2}$$

with appropriate normalization in the case of integration over the group volume with measure

$$\frac{(n-1)!}{2\pi^n}\prod_{r=2}^{n}\sin^{2r-3}\theta_r\cos\theta_r\,\mathrm{d}\theta_r\prod_{i=1}^{n}\mathrm{d}\varphi_i.$$

Here $D^{n-1[p',\dot{0},-q']}_{Q_{(n-1)};0}(g')$ are the matrix elements of $U(n-1)$ irrep $[p',\dot{0},-q'] = [p_{(n-1)},\dot{0},-q_{(n-1)}]$ (with parameters obtained after omitting φ_n and θ_n. Special matrix elements of the $U(r)$ irreducible representation $[p,\dot{0},-q]$ with the $U(r-1)$ irrep labels $[p',\dot{0},-q']$ and 0 and $SU(r-2)$ irrep label 0 for rotation with angle θ_r in the (x_r, x_{r-1}) plane are written in terms of the D-matrices of $SU(2)$ as follows:

$$D^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_r) = \left[(p+q+r-1)d^{(r-1)}_{[p',\dot{0},-q']}\right]^{1/2}\left[(r-1)d^{(r)}_{[p,\dot{0},-q]}\right]^{-1/2} \times (\mathrm{i}\sin\theta_r)^{-r+2}P^{(p+q+r-2)/2}_{p'+(q-p+r-2)/2,-(p-q+r-2)/2-q'}(\cos 2\theta_r) \tag{6.3}$$

and further, taking into account the identity $P^l_{m,n}(x) = P^l_{-n,-m}(x)$, in terms of the Jacobi polynomials

$$D^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_r) = \mathcal{N}^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']}\left[d^{(r-1)}_{[p',\dot{0},-q']}\left((r-1)d^{(r)}_{[p,\dot{0},-q]}\right)^{-1}\right]^{1/2} \times (\mathrm{i}\sin\theta_r)^{p'+q'}(\cos\theta_r)^{|M|}P^{(L'+r-2,|M|)}_K(\cos 2\theta_r) \tag{6.4}$$

where

$$K = \min(p-p', q-q') \qquad M = p-q-p'+q' \qquad L' = p'+q'$$

and

$$\mathcal{N}^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']} = \left[\frac{(p+q+r-1)K!(p+q+r-2-K)!}{(|M|+K)!(p+q+r-2-|M|-K)!}\right]^{1/2} \tag{6.5a}$$

$$= \left[\frac{(p+q+r-1)K!(L'+r-2+|M|+K)!}{(|M|+K)!(L'+r-2+K)!}\right]^{1/2}. \tag{6.5b}$$

Factor $\mathrm{i}^{p'+q'}$, also appeared in [41] (but was absent in the generic expressions of D-matrix elements [36, 42]), ensures the complex conjugation relation

$$\overline{D^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_r)} = (-1)^{p'+q'}D^{r[q,\dot{0},-p]}_{[q',\dot{0},-p']0;0}(\theta_r) \tag{6.6}$$

in accordance with the $SU(2)$ case and the system of phases of Baird and Biedenharn [43], which is correlated to the positive signs of the Gel'fand–Tsetlin matrix elements [35, 36, 44] of the $U(n)$ generators $E_{r,r-1}$. Alternatively, the states $\Psi_{p'+q',p+q,M}(\theta_r)$, as defined in [4, 40] and related to the hyperspherical harmonics, correspond to the Jacobi polynomials with interchanged parameters α and β. Hence the variables are mutually reflected (here and in [4, 40] as $\cos 2\theta_r$ and $(-\cos 2\theta_r)$).

Using the integration over group (cf [36, 40, 45]), the corresponding $3j$-symbols of the class-two irreps for the chain $U(n) \supset U(n-1)\times U(1) \supset \cdots \supset U(2)\times U(1) \supset U(1)$ may be factorized as follows:

$$\sum_\rho \begin{pmatrix} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p_3,\dot{0},-q_3] \\ Q_{1(n)} & Q_{2(n)} & Q_{3(n)} \end{pmatrix}^\rho_n \times \begin{pmatrix} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p_3,\dot{0},-q_3] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{pmatrix}^\rho_n = \int_{U(n)} \mathrm{d}g\, D^{n[p_1,\dot{0},-q_1]}_{Q_{1(n)};0}(g)D^{n[p_2,\dot{0},-q_2]}_{Q_{2(n)};0}(g)D^{n[p_3,\dot{0},-q_3]}_{Q_{3(n)};0}(g) \tag{6.7a}$$

$$
\begin{aligned}
&= \delta_{p_1+p_2+p_3,q_1+q_2+q_3}(-1)^{p_1'+p_2'+p_3'+K_1+K_2+K_3}(n-1)^{-1/2} \\
&\quad\times \prod_{a=1}^{3} \mathcal{N}^{n[p_a,\dot{0},-q_a]}_{[p_a',\dot{0},-q_a']} \left[d^{(n-1)}_{[p_a',\dot{0},-q_a']} \left(d^{(n)}_{[p_a,\dot{0},-q_a]} \right)^{-1} \right]^{1/2} \\
&\quad\times \widetilde{\mathcal{I}} \begin{bmatrix} 0, n-2 & |M_1|, L_1'+n-2 & |M_2|, L_2'+n-2 & |M_3|, L_3'+n-2 \\ & K_1 & K_2 & K_3 \end{bmatrix} \\
&\quad\times \sum_{\rho'} \begin{pmatrix} [p_1', \dot{0}, -q_1'] & [p_2', \dot{0}, -q_2'] & [p_3', \dot{0}, -q_3'] \\ Q'_{1(n)} & Q'_{2(n)} & Q'_{3(n)} \end{pmatrix}^{\rho'}_{n-1} \\
&\quad\times \begin{pmatrix} [p_1', \dot{0}, -q_1'] & [p_2', \dot{0}, -q_2'] & [p_3', \dot{0}, -q_3'] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{pmatrix}^{\rho'}_{n-1}. \qquad (6.7b)
\end{aligned}
$$

Here ρ and ρ' are the multiplicity labels of the $U(n)$ and $U(n-1)$ scalars in the decompositions $[p_1, \dot{0}, -q_1] \otimes [p_2, \dot{0}, -q_2] \otimes [p_3, \dot{0}, -q_3]$ and $[p_1', \dot{0}, -q_1'] \otimes [p_2', \dot{0}, -q_2'] \otimes [p_3', \dot{0}, -q_3']$. The integral involving the product of three Jacobi polynomials that appeared in (6.7b) also corresponds to the $SO(2n) \supset SO(2n-2) \times SO(2)$ isofactor of $3j$-symbol

$$
\begin{pmatrix} p_1+q_1 & p_2+q_2 & p_3+q_3 \\ p_1'+q_1', M_1 & p_2'+q_2' M_2 & p_3'+q_3', M_3 \end{pmatrix}_{(2n:2n-2,2)}
$$

considered in the previous section and may be expressed (after some permutations of the parameters) as a double sum by means of (3.3a) or (3.3b). For normalization of the corresponding $3j$-symbols of $U(n) \supset U(n-1)$ we may use the square root of

$$
\begin{aligned}
&\sum_{\rho} \left[\begin{pmatrix} [p_1, \dot{0}, -q_1] & [p_2, \dot{0}, -q_2] & [p_3, \dot{0}, -q_3] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{pmatrix}^{\rho}_{n} \right]^2 = \delta_{p_1+p_2+p_3,q_1+q_2+q_3} \\
&\quad\times (-1)^{\min(p_1,q_1)+\min(p_2,q_2)+\min(p_3,q_3)} \frac{(n-1)![(n-2)!]^2}{\prod_{a=1}^{3} (\min(p_a,q_a)+1)_{n-2}} \\
&\quad\times \widetilde{\mathcal{I}} \begin{bmatrix} 0, n-2 & |p_1-q_1|, n-2 & |p_2-q_2|, n-2 & |p_3-q_3|, n-2 \\ & \min(p_1,q_1) & \min(p_2,q_2) & \min(p_3,q_3) \end{bmatrix} \qquad (6.8)
\end{aligned}
$$

with non-vanishing extreme $3j$-symbols on the left-hand side for a single value of the multiplicity label ρ, which is not correlated to the canonical [46–48] and other (see [49–51]) external labelling schemata of the coupling coefficients of $U(n)$. In contrast to the particular $3j$-symbols (4.9) of $SO(n)$, equation (6.8) is summable only in the multiplicity-free cases. In addition to three double-sum versions of (3.3a) and (3.3b), integral on the right-hand side of (6.8) may also be expressed as three different double-sum series by means of (3.2e), taking into account the symmetry relation (3.2b). Of course, (6.8) is always positive as an analogue of the denominator function of the $SU(3)$ canonical tensor operators [46–48, 52].

Taking into account (6.6) we may also obtain an expression for the Clebsch–Gordan coefficients of the class-two representation of $U(n)$

$$
\begin{aligned}
&\sum_{\rho} \left[\begin{matrix} [p_1, \dot{0}, -q_1] & [p_2, \dot{0}, -q_2] \\ Q_{1(n)} & Q_{2(n)} \end{matrix} \middle| \begin{matrix} [p, \dot{0}, -q] \\ Q_{(n)} \end{matrix} \right]^{\rho}_{n} \\
&\quad\times \left[\begin{matrix} [p_1, \dot{0}, -q_1] & [p_2, \dot{0}, -q_2] \\ [\dot{0}] & [\dot{0}] \end{matrix} \middle| \begin{matrix} [p, \dot{0}, -q] \\ [\dot{0}] \end{matrix} \right]^{\rho}_{n} \\
&= d^{(n)}_{[p,\dot{0},-q]} \int_{U(n)} \mathrm{d}g\, D^{n[p_1,\dot{0},-q_1]}_{Q_{1(n)};0}(g)\, D^{n[p_2,\dot{0},-q_2]}_{Q_{2(n)};0}(g)\, \overline{D^{n[p,\dot{0},-q]}_{Q_{(n)};0}(g)} \qquad (6.9)
\end{aligned}
$$

with the integrals involving the product of three Jacobi polynomials and the CG coefficients of $U(n-1)$ of the same type and some phase and irrep dimension factors. In particular, we obtain the following expression for isofactors of special $SU(3)$ Clebsch–Gordan coefficients (which perform the coupling of the $SU(3)$-hyperspherical harmonics):

$$\begin{bmatrix} (a'b') & (a''b'') & (ab)_0 \\ (z')i' & (z'')i'; & (z)i \end{bmatrix}$$

$$= \delta_{a'+a''-a,b'+b''-b}(-1)^{i'+i''-i+K_1+K_2-K}\frac{1}{2}\left[\frac{(2i'+1)(2i''+1)}{(2i+1)d^{(3)}_{(a'b')}d^{(3)}_{(a''b'')}}\right]^{1/2}$$
$$\times\left\{\frac{(-1)^{\min(a',b')+\min(a'',b'')+\min(a,b)}}{(\min(a',b')+1)\,(\min(a'',b'')+1)\,(\min(a,b)+1)}\right.$$
$$\left.\times\,\widetilde{\mathcal{I}}\begin{bmatrix}0,1 & |a'-b'|,1 & |a''-b''|,1 & |a-b|,1 \\ & \min(a',b') & \min(a'',b'') & \min(a,b)\end{bmatrix}\right\}^{-1/2}$$
$$\times\,\mathcal{N}^{3[a',0,-b']}_{[i'-z',-i'-z']}\mathcal{N}^{3[a'',0,-b'']}_{[i''-z'',-i''-z'']}\mathcal{N}^{3[a,0,-b]}_{[i-z,-i-z]}\begin{bmatrix} i' & i'' & i \\ z' & z'' & z\end{bmatrix}$$
$$\times\,\widetilde{\mathcal{I}}\begin{bmatrix}0,1 & |M'|,2i'+1 & |M''|,2i''+1 & |M|,2i+1 \\ & K' & K'' & K\end{bmatrix}. \qquad (6.10)$$

Here $M = a-b+2z$, $K = \min(a+z-i, b-z-i)$ in the notation of [49, 51], with (ab) for the mixed tensor irreps, where $a = p_{(3)}, b = q_{(3)}$ and the basis states are labelled by the isospin $i = \frac{1}{2}(p_{(2)} + q_{(2)})$, its projection $i_z = p_{(1)} - \frac{1}{2}(p_{(2)} - q_{(2)})$ and the parameter $z = \frac{1}{3}(b-a) - \frac{1}{2}y = \frac{1}{2}(q_{(2)} - p_{(2)})$ instead of the hypercharge $y = p_{(2)} - q_{(2)} - \frac{2}{3}(p_{(3)} - q_{(3)})$.

7. Concluding remarks

In this paper, we reconsidered the $3j$-symbols and Clebsch–Gordan coefficients of the orthogonal $SO(n)$ and unitary $U(n)$ groups for all three representations corresponding to the (ultra)spherical or hyperspherical harmonics of these groups (i.e. irreps induced [35] by the scalar representations of the $SO(n-1)$ and $U(n-1)$ subgroups, respectively). For the corresponding isoscalar factors of the $3j$-symbols and coupling coefficients, the ordinary integrations involving triplets of the Gegenbauer and the Jacobi polynomials yield the most symmetric triple-sum expressions, however, without the apparent triangle conditions. These conditions are visible and efficient only in expressions of the type [6, 15] derived after complicated analytical continuation procedures of special $Sp(4) \supset SU(2) \times SU(2)$ isofactors (cf [14, 17]). Actually, only for a fixed integer shift parameter $p_i = \frac{1}{2}(l_j + l_k - l_i)$ is it evident that the corresponding integrals involving triplets of the Gegenbauer and the Jacobi polynomials are rational functions of remaining parameters. Practically, the concept of the canonical unit tensor operators (see section 21 of chapter 3 of [26]) for symmetric irreps of $SO(n)$ may be formulated only under such a condition.

Similarly as special terminating double-hypergeometric series of Kampé de Fériet-type [19–21, 53] correspond to the stretched $9j$ coefficients of $SU(2)$, the definite terminating triple-hypergeometric series correspond either to the semistretched isofactors of the second kind [14] of $Sp(4)$ or to the isofactors of the symmetric irreps of the orthogonal group $SO(n)$ in the canonical and semicanonical (tree-type) bases. Our relation (2.6*a*)–(2.6*c*) (which is significant within the framework of $Sp(4)$ isofactors) is a triple-sum generalization of transformation formula (9) of [21] for terminating $F^{1:2;2}_{1:1;1}$ Kampé de Fériet series with a fixed single-integer non-positive parameter, restricting all summation parameters. (This restriction is hidden in equations (2.7*a*) and (2.7*b*), and rearranged for the aims of section 3.) Relations (2.6*a*)–(2.6*c*), with intermediate formula (2.6*b*), were derived using the transformation formulae [19, 21] of the double series, treated as stretched $9j$ coefficients. The relation (3.2*c*)–(3.2*e*) (important within the framework of $SO(n)$ isofactors) cannot be associated with any transformation formula [21] for terminating $F^{1:2,2}_{1:1,1}$ Kampé de Fériet series with the same (single or double) parameters, restricting summation.

The wish may arise to prove identity (3.2*c*)–(3.2*e*) by a direct transformation procedure, without using the auxiliary rearrangement of section 2. Initially, the relation between (3.2*c*) and analytical continuation of (2.6*b*) (with the same three parameters restricting summation in the both cases) may be proved, using composition of transformation formulae of section 3 of [21] for the double sum over z_j and z_k in (3.2*c*) as terminating $F^{1:2,2}_{1:1,1}$ series into terminating $F^{1:2,2}_{0:2,2}$ series. Furthermore, the double-sum version of relation between (3.2*e*) (e.g. for $k_k = z_k = 0$) and the result of the previous step need to be considered. Both the sums over z_l (for α_0 and β_0 integers) can be recognized in the same Clebsch–Gordan coefficient of $SU(2)$. Hence, the identity between two ${}_3F_2(1)$ series [28, 29, 54] (for arbitrary α_0 and β_0) induces the identity between the terminating double series. In its turn, inserting the latter one induces the identity between the terminating triple series, in which parameters restricting summation coincide only in part. Direct transformation of single ${}_3F_2(1)$ series [28, 29, 54] is useless for the proof of identities (2.6*a*)–(2.6*c*) and (3.2*c*)–(3.2*e*).

Expressions (2.6*a*) and (2.6*c*) corresponding to special $Sp(4)$ isofactors are summable or turn into the terminating Kampé de Fériet [20, 21] series $F^{2:2}_{2:1}$ for extreme basis states of $Sp(4) \supset SU(2) \times SU(2)$. Alternatively, in accordance with (3.6*c*) and (3.2*e*), the expressions for special isofactors of $SO(n)$ and $SU(n)$ are summable in the case of the stretched couplings of the group representations and turn into the terminating Kampé de Fériet series $F^{2:2}_{2:1}$ for the irreps of subgroups in a stretched situation, including the generic cases for the restrictions $SO(n) \supset SO(n-1)$, $SO(n) \supset SO(n-2) \times SO(2)$ and $U(n) \supset U(n-1)$. Taking into account the fact that the $F^{2:2}_{2:1}$-type series with five independent parameters also appeared as the denominator (normalization) functions of the $SU(3)$ and $u_q(3)$ canonical tensor operators [46–48] (cf (2.8) and section 2 of [52]), the q-extension of relation (3.3*a*)–(3.3*b*) from the classical $SU(n)$ case may be suspected.

In a following paper, the fourfold [16, 30] and (corrected) [16] triple-sum expressions for the recoupling ($6l$) coefficients of symmetric irreps of $SO(n)$ will be rearranged into double $F^{1:4}_{1:3}$-type series.

Appendix A. Special cases of triple sums in 11j coefficients

Rearranging (3.12) in the inverse order that was used for the transition from (2.6*a*) through (2.7*a*) to (3.2*c*), we derived an expression for a special triple sum of the type (2.2) with the coinciding two first rows of the corresponding array,

$$\widetilde{\mathbf{S}}\begin{bmatrix} K_1 & j_1^1 & j_1^2 & j_1^3 \\ K_1 & j_1^1 & j_1^2 & j_1^3 \\ 2K_1 & j^1 & j^2 & j^3 \end{bmatrix}$$

$$= (-1)^{j_1^1+j_1^2+j_1^3-K_1-j^3}[1+(-1)^{j^1+j^2+j^3-2K_1}]2^{j^1+j^2-2K_1-1}(2j^3-1)!j^3!$$

$$\times \prod_{a=1}^{2} \frac{\left(2j_1^a+j^a+1\right)!(j^a!)^2}{(2j^a+1)!\left(2j_1^a-j^a\right)!} \sum_{x_1,x_2,z_3} \binom{j_1^3-\frac{1}{2}(j^3+\delta_3)+x_3}{x_3}$$

$$\times \frac{(-1)^{x_3}\left(-j_1^3-(j^3+\delta_3)/2\right)_{x_3}}{(-j^3+1/2)_{x_3}}\begin{pmatrix}\frac{1}{2}(\delta_1+\delta_2-\delta_3)\\ K_1+j^3-\sum_{a=1}^{3}\left(\frac{1}{2}j^a-x_a\right)\end{pmatrix}$$
$$\times\prod_{a=1}^{2}\frac{\left(j_1^a+(j^a+\delta_a)/2+1\right)_{x_a}}{(j^a+3/2)_{x_a}}\begin{pmatrix}j_1^a-\frac{1}{2}(j^a+\delta_a)\\ x_a\end{pmatrix} \tag{A.1}$$

with $\delta_i = 0$ or 1 such that $j_1^i - (j^i + \delta_i)/2$ $(i = 1, 2, 3)$ are integers.

Furthermore, similarly rearranging (3.13), we derived an expression for a special triple sum $\widetilde{\mathcal{S}}[\cdots]$ of the type (2.2) with the coinciding two first rows and $j_1^1 = j_1^2$, $j_1^3 = K_1$:

$$\widetilde{\mathrm{S}}\begin{bmatrix} j_1^3 & j_1^1 & j_1^1 & j_1^3\\ j_1^3 & j_1^1 & j_1^1 & j_1^3\\ 2j_1^3 & j^1 & j^2 & j^3\end{bmatrix}$$
$$=\frac{\left(2j_1^3-2j_1^1\right)!\Gamma(1/2)\Gamma\left((j^1+j^2+j^3+1)/2-j_1^3\right)\prod_{a=1}^{3}j^a!}{2^{4j_1^3+3}\prod_{a=1}^{3}\Gamma\left(j_1^3+(j^1+j^2+j^3+3)/2-j^a\right)}$$
$$\times\sum_s\frac{\left(2j_1^1+j^1+1\right)!\left(2j_1^1+j^2+1\right)!\left(2j_1^3+j^3+1\right)!}{s!\left(2j_1^3-2j_1^1-s\right)!\left(j_1^3+(j^1-j^2-j^3)/2-s\right)!}$$
$$\times\frac{[1+(-1)^{j^1+j^2+j^3-2j_1^3}](-1)^{2j_1^1+j_1^3-(j^1+j^2+j^3)/2}}{\left(j_1^3+(j^2-j^3-j^1)/2-s\right)!\left(2j_1^1-j_1^3+(j^3-j^1-j^2)/2+s\right)!}$$
$$\times\frac{\left(2j_1^1+1/2\right)_s\Gamma\left(2j_1^3+3/2-s\right)}{\left(2j_1^1-j_1^3+(j^1+j^2+j^3)/2+s+1\right)!}. \tag{A.2}$$

Although this sum again corresponds to the balanced (Saalschützian) ${}_4F_3(1)$-type series [28, 29], it is not alternating (since it includes even numbers of gamma functions or factorials in the numerator and the denominator) and cannot be associated with the $6j$ coefficients of $SU(2)$. Note that the summable case of (A.2) with $j_1^3 = j_1^1$ corresponds to (41) of [14].

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